

# Towards a formalization of a two traders market with information exchange

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## Abstract

This paper shows that Hamiltonians and operators can also be put to good use even in contexts which are not purely physics based. Consider the world of finance. The work presented here models a two traders system with information exchange with the help of four fundamental operators: cash and share operators; a portfolio operator and an operator reflecting the loss of information. An information Hamiltonian is considered and an additional Hamiltonian is presented which reflects the dynamics of selling/buying shares between traders. An important result of the paper is that when the information Hamiltonian is zero, portfolio operators commute with the Hamiltonian and this suggests that the dynamics are really due to the information. Under the assumption that the interaction and information terms in the Hamiltonian have similar strength, a perturbation scheme is considered on the interaction parameter. Contrary to intuition, the paper shows that up to a second order in the interaction parameter, a key factor in the computation of the portfolios of traders will be the initial values of the loss of information (rather than the initial conditions on the cash and shares). Finally, the paper shows that a natural outcome from the inequality of the variation of the portfolio of trader one versus the variation of the portfolio of trader two, begs for the introduction of ‘good’ and ‘bad’ information. It is shown that ‘good’ information is related to the reservoirs (where an infinite set of bosonic operators are used) which model

rumors/news and external facts, whilst ‘bad’ information is associated with a set of two modes bosonic operators.

# I Motivations

The application of techniques from physics to areas outside of its natural remit, such as economics and finance, is not new. In the 1960's the famous Harvard economist Nicolas Georgescu-Roegen [14] considered the concept of entropy in economics. In the 1990's the econophysics movement started by luminaries like Eugene Stanley [23] and Bouchaud [10], became an important field where techniques from statistical mechanics were very fruitful in understanding some difficult problems in finance and macro-economics. An outstanding paper which argues in a very informed way about how physics based ideas can be of benefit to understanding a plethora of concepts in the area of complex systems is by Kwapień and Drożdż [22]. In that paper the authors remark that one could guess, from a theoretical point of view that “all the laws of financial economics must be a strict mathematical consequence of the four fundamental interactions among the elementary particles”. But the authors do immediately caution, that from a **practical** point of view such an approach is unworkable. The authors remark that “to fully explain the financial market's behavior, one has to neglect the deeper levels of organization without any meaningful loss of information”. Kwapień and Drożdż [22] ask the question, within the context of real stock data, what part of the eigenvalue spectrum of the correlation matrix (which contains a Wishart matrix) contains information about non-trivial correlations. The authors find that, again within the context of stock markets, noise and collectivity (i.e. based on a large number of non-linear interacting constituents) are in a dynamical balance with each other, and this typifies complex systems. The rationale for introducing some ideas out of quantum mechanics is also highlighted in the same paper [22], where the authors do mention what the interpretation might be of a price of an asset between two consecutive transactions. Indeed the non-observed price could be associated with a quantum mechanical measurement problem. The authors cite the work of Schaden [30] in that regard. Other works appeared also in that area. In the field of game theory, the solution space of even very elementary games can be enriched when a quantum mechanical interpretation is considered. In the paper by Piotrowski and Ślaskowski [26] the authors invite the reader to consider what happens when trading strategies are allowed to be entangled. The paper proposes the idea of a quantum strategy. An important paper, again by these authors [27], investigates a crucial aspect for improving our understanding of

financial markets: Information. The concept of information is well formalized in physics and the paper shows that the formalization of information via the metric structures can be a very good step in the right direction for a deeper theoretical understanding of financial markets. In effect we will below, in the motivation for this paper, address the information issue a little more.

The area of research which applies techniques from quantum mechanics to a variety of problems in the social sciences, can actually be traced back to the 1950's, with the discussions physics Nobelist Pauli had with the well known psychologist Jung, on basic issues such as how complementarity in quantum physics can have 'some' existence in psychology [24], [21]. The level of effectiveness by which quantum mechanical techniques have been able to shed further light on thorny problems in a variety of areas in the social sciences, varies somewhat. In psychology, there is sizable research-momentum in the particular field which actively uses probability interference to decision making paradoxes in economics and psychology [11], [13]. In the area of information retrieval, research advances are also made [12]. In finance, which is the area of application of this current paper, progress has been made on importing the quantum physical machinery in an attempt to augment the modelling of information. Other work in this area has also looked at how potential functions (within the quantum mechanical setting) can adopt financial meaning [32], [2].

In a recent paper, [3], the authors have considered the way in which information reaching different traders of a (simplified) stock market influences the behavior of the traders, **before they begin to trade**. In other words, we have considered what happens before the market opens, and in which way the strategy of the traders is generated. In this description we have used tools which are originally encountered in the microscopic world, and which have been proven to be useful also in the description of different classical systems, see [5] for a recent review. In particular, a special role is played by an operator, the Hamiltonian of the system, which is used to deduce the dynamics of those quantities we are interested in, the so-called *observables* of the model.

In some older papers of one of us (F.B.), [6]-[9], the role of information was, in a certain sense, simply incorporated by properly choosing some of the constants defining the Hamiltonian of the system we were considering. The Hamiltonian is adopted to mimic and describe the interactions between the traders, [5]. On the other hand, E.H. and his

coworkers, following the original idea of [20], considered the role of information for stock markets, [15]-[17], mainly adopting the Bohm view of quantum mechanics, where the information is carried by a pilot wave function  $\Psi(x, t)$ , satisfying a certain Schrödinger equation of motion, and which, with simple computations, produces what in the literature is called a *mental force*. This force has to be added to the other *hard* forces acting on the system, producing a full Newton-like classical differential equation.

In [3] we have tried to produce an unifying point of view, using Bohmian quantum mechanics to construct a Hamiltonian  $H$  in which the information is not merely described by some parameters of  $H$ , but becomes one of the dynamical variables of the system. However, in that preliminary work, we have only considered how the information contributes to generate, out of two *equivalent* traders  $\tau_1$  and  $\tau_2$ , two traders which are no longer equivalent: i.e. they have used the information to improve, as much as they can, their *financial status* (the portfolio, see below). For this reason, no interaction between  $\tau_1$  and  $\tau_2$  was considered in [3]. Here we continue our analysis adding also a possible interaction to the system. In other words, we will see what happens in a market made of  $\tau_1$  and  $\tau_2$ , when they interact and are also subjected to a flux of information coming from the market itself and from the outer world. As one can expect, this is quite a hard problem to be discussed in its full generality, and in fact we will consider, along the way, some useful assumptions which will allow us to deduce an approximate analytical solution for the problem.

To clarify our main ideas, we propose here a list of six succinct points which are those motivating our present analysis. We keep specifically in mind the ‘un-convinced’ or ‘sceptic’ reader.

- First, the Hamiltonians which are used in the paper are introducing dynamics in the model in a “natural” way. We can explicitly claim that the Hamiltonians considered here are receiving an economics based interpretation. Important work in the literature has also referred to the use of a Hamiltonian framework in a social science framework. In Kwapien and Drożdż [22] reference is made to a so called market factor which is a force acting on all stocks. As the authors explain, this approach refers to a many-body problem which can lead to the use of a Hamiltonian. In their paper Piotrowski and Sładkowski [28] use a Hamiltonian which contains what

they define as a ‘risk inclination operator’. Our paper expands the Hamiltonian (relative to our first paper (Bagarello and Haven [3])) to a Hamiltonian which now also models interaction, even if in a very simplified form. Whilst our first paper had an absence of interaction between traders, and this current paper explicitly allows for interaction between traders, it should be stressed that **even in the absence of interaction** there was quite some richness in the first paper. The limit on number of traders was of course irrelevant given the absence of interaction, but even with this absence, we were very concerned to discuss what happens before trading begins and after the rumors have reached traders. Our first paper also actively studied the situation of two traders who are no longer completely equivalent. In this paper, we think that it is quite important to observe that we can now divide information, using the expanded Hamiltonian framework, into two sets of information - bad and good information. Information is seen as a dynamical variable and it thus has a role in the Hamiltonian itself. This leads us to make a plea about how useful in fact quantum mechanical concepts in social science can be. We believe that the modelling of information is a very big advantage that the quantum formalism has to offer when considering applications outside of the remit of quantum mechanics. We want to hint to the use of Fisher information (well known in economics via the so called Cramer-Rao bound) and the intimate relationship which exists between the minimization of Fisher information and the Schrödinger equation (see Hawkins and Frieden [19]). Please see also point five below. We also can mention the relationship which has been argued for between Fisher information and a specific type of potential (see Reginatto [29], Haven and Khrennikov [18]).

- Second, the use of non-commuting operators has been investigated in the finance environment. In Segal and Segal [31] it is shown that such operators should be used to describe the time dependence of the price of shares and its forward time derivative. The motivation is purely economical: if one trader knows exactly both these quantities, he could earn a virtually enormous amount of money. Since this does not happen, it is reasonable to replace functions of time with time-dependent, non commuting, operators.

- Third, bosonic operators have a financial meaning in this paper and the reason why such bosonic operators are coming in a natural way in our financial set up is linked to the fact that the operator can assume a very large set of discrete values. This gives us the possibility to describe, in a rather natural way, the portfolios (see below) of the traders.
- Fourth, the reservoir with which the traders interact produces a system with infinite degrees of freedom.
- Fifth, we can, as we have expressed in several footnotes in the current paper, look at the measure of loss of information within the context of a traded financial payoff function and Fisher information (which we mentioned in our first point above).
- Sixth, we probably should also mention that very strong connections have been established between the Schrödinger and the Black-Scholes equations, [1]. This is surely another indication of the relevance of quantum mechanics in economics.

In summary, we think both quantum mechanics and financial markets benefit from our approach. From a quantum mechanical point of view, we show that uses can be made of elementary concepts outside of the natural remit of quantum mechanics. We believe that the above 6 points provide for good arguments why this current study can provide benefits for better understanding financial markets. Very few models in economics will use Hamiltonians which have an information and interaction component to describe dynamics. We can only make ‘baby-steps’ at this point in time, but it should be seen as a credible argument, that given the extremely powerful machinery quantum mechanics really is, it may not be impossible to harness that power also within a social science domain. It is surely not the case, that finance and economics should not be receptive to new models. Quite the contrary, for its own survival, it should be open to models coming from other areas of inquiry. Many models in the finance literature are often extremely simple too. Often the assumptions underlying those models make the applicability of the model to be very constrained. We have been very up-front in this paper with our assumptions.

The paper is organized as follows: in the next section we propose our model and we discuss some of its most important aspects. In particular, we deduce the relevant

equations of motion. In Section III we propose a perturbative approach to deduce the approximate solution of these equations. Section IV contains our conclusions.

## II The model

The model we are interested here extends the one originally proposed in [3], adding an explicit interaction term between the traders. We begin by defining the following Hamiltonian, already considered in [3]:

$$\begin{cases} \mathfrak{H} = H_0 + H_{inf}, \\ H_0 = \sum_{j=1}^2 \left[ \omega_j^s \hat{S}_j + \omega_j^c \hat{K}_j + \Omega_j \hat{I}_j + \int_{\mathbb{R}} \Omega_j^{(r)}(k) \hat{R}_j(k) dk \right], \\ H_{inf} = \sum_{j=1}^2 \left[ \lambda_{inf} \left( i_j (s_j^\dagger + c_j^\dagger) + i_j^\dagger (s_j + c_j) \right) + \gamma_j \int_{\mathbb{R}} (i_j^\dagger r_j(k) + i_j r_j^\dagger(k)) dk \right], \end{cases} \quad (2.1)$$

where  $\hat{R}_j(k) = r_j^\dagger(k) r_j(k)$ ,  $\hat{S}_j = s_j^\dagger s_j$ ,  $\hat{K}_j = c_j^\dagger c_j$  and  $\hat{I}_j = i_j^\dagger i_j$ , and the following canonical commutation relations (CCR's) are assumed,

$$[s_j, s_l^\dagger] = [c_j, c_l^\dagger] = [i_j, i_l^\dagger] = \mathbb{1} \delta_{j,l}, \quad [r_j(k), r_l^\dagger(q)] = \mathbb{1} \delta_{j,l} \delta(k - q), \quad (2.2)$$

all the other commutators being zero. Moreover  $\omega_j^s$ ,  $\omega_j^c$ ,  $\Omega_j$ ,  $\lambda_{inf}$  and  $\gamma_j$  are real constants, while  $\Omega_j^{(r)}(k)$ ,  $j = 1, 2$ , are two real-valued functions. Each bosonic operator has a different meaning in the present context, which is explained in detail in [3]:  $c_j$ ,  $c_j^\dagger$  and  $\hat{K}_j$  are *cash operators*. They respectively lower, increase and count the units of cash in the portfolio of  $\tau_j$ , see below. Analogously  $s_j$ ,  $s_j^\dagger$  and  $\hat{S}_j$  are *share operators*. They lower, increase and count the number of shares in the portfolio of  $\tau_j$ . Incidentally, we notice that, to make the notation simple, we are assuming that our market consists of a single type of shares. This is not a major constraint, and it could be avoided. However, we will not do it here. The operator  $i_j^\dagger$  increases the *lack of information* (LoI) of  $\tau_j$ , while  $i_j$  decreases it. Of course, the higher the value of the eigenvalues of the number-like operator  $\hat{I}_j$ , the less  $\tau_j$  knows about what is going on in the market. In other words, to be more efficient, the trader should have a low LoI, i.e. he should be somehow associated to a small eigenvalue of  $\hat{I}_j$ . In our model we also have a reservoir, which models the set of all the rumors,



news, and external facts which, all together, concretely create the final information and, therefore, fix the values of the LoI's of the two traders. The reservoir<sup>1</sup> is described here by the bosonic operators  $r_j(k)$ ,  $r_j^\dagger(k)$  and  $\hat{R}_j(k)$ , which depend on a real variable,  $k \in \mathbb{R}$ .

The Hamiltonian  $\mathfrak{H}$  contains a free *canonical* part  $H_0$ . By this we mean that  $H_0$  is the typical quadratic Hamiltonian used in quantum many-body systems, when they are described in second quantization. The main characteristic of  $H_0$  is that, whenever our system is described only by  $H_0$ , i.e. when we put  $H_{inf} = 0$ , all the number operators ( $\hat{S}_j$ ,  $\hat{K}_j$  and so on) stay constant in time: so, from the point of view of our *observables*, [5], the market looks static. However, this is not really so, since non-observable operators may still evolve in time.

For what concerns  $H_{inf}$ , let us now consider separately its two contributions. They respectively describe the following: when the LoI increases, the value of the portfolio decreases (because of  $i_j^\dagger(s_j + c_j)$ ) and vice-versa (because of  $i_j(s_j^\dagger + c_j^\dagger)$ )<sup>2</sup>. Moreover, the LoI increases when the "value" of the reservoir decreases (this is the meaning of  $i_j^\dagger r_j(k)$ ), and, viceversa, decreases when the "value" of the reservoir increases<sup>3</sup>. Considering, for

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<sup>1</sup>In financial economics, a distinction is often made between so called 'private information' and 'public information'. The reservoir here contains public and private information. Distinguishing those types of information can be fruitful as they do implicitly call up notions such as 'financial efficiency' where the strongest form of efficiency would say that all prices contain both public and private information. This is thus the point of view taken in this paper.

<sup>2</sup>This first contribution to  $H_{inf}$  can also be obtained via a slightly different route ([16]) where a quantum mechanical (like) wave function is seen as carrier of information and upon it travelling towards a potential function (the payoff function) it may decay or not depending on the position of the potential versus total energy. Total energy is considered as capturing public information, whilst the wave function carries private information. If we assume the portfolio to be a payoff function which maps a domain of prices of shares onto a level of profit then one can model incoming information, before a profit position is taken, as decaying in a way which will depend on the level of the profit. We show that if we restrict this domain of prices to be very narrow, then the higher the level of profit of the payoff function the lower the LoI and the lower the level of profit, the higher the LoI. The change in LoI could be measured via the comparison of two Fisher information measures.

<sup>3</sup>If we consider again [16], we can obtain a similar result - but again in a different setting. Consider the reservoir to be total energy and let there be a payoff function (which is a potential function) with a large domain of prices of shares. Set first the level of total energy vis a vis the payoff function such that the incoming quantum (like) mechanical wave function does not decay and calculate the Fisher information. Now reduce the level of total energy such that the incoming quantum (like) mechanical wave function

example, the contribution  $i_j r_j^\dagger(k)$  in  $H_{inf}$ , we see that the LoI decreases (so that the trader is *better informed*) when a larger amount of news, rumors, etc. reaches the trader. Notice that, in  $\mathfrak{H}$ , no interaction between  $\tau_1$  and  $\tau_2$  is considered, yet. As in [3], to produce a reasonably simple model, we will assume that the price of the share is constant in time, and we fix this constant to be one. Of course, this is a strong limitation of the model, but it is useful to allow to get some analytical expression for the time evolution of the portfolios of the traders. Other possibilities exist, but, not surprisingly, produce more complicated models: one could consider the price of the share as a dynamical variable of the system. This is what we really would like to do, but it is very hard to implement this possibility in a realistic way. A simpler possibility is to consider the price as an external field, deduced out of experimental data. Both these possibilities are discussed in [5]. We will come back on this aspect of the model later on.

In [3]  $\mathfrak{H}$  was exactly the objective of our interest, since we were not considering the interaction between the traders. Here, on the other hand, this is exactly one of the aspects which is interesting for us. For this reason, our full model is described by the following Hamiltonian:

$$\begin{cases} H = \mathfrak{H} + H_{int}, \\ H_{int} = \lambda \left( s_1 c_1^\dagger s_2^\dagger c_2 + s_1^\dagger c_1 s_2 c_2^\dagger \right). \end{cases} \quad (2.3)$$

The meaning of  $H_{int}$  is the following:  $s_1 c_1^\dagger s_2^\dagger c_2$  describes the fact that  $\tau_1$  is selling a share to  $\tau_2$ . For this reason, the number of the shares in his portfolio decreases of one unit (and this is the meaning of  $s_1$ ) while his cash increases of one unit (because of  $c_1^\dagger$ ), since the price of the share is assumed here to be one<sup>4</sup>. After the interaction,  $\tau_2$  has one more share ( $s_2^\dagger$ ), but one less unit of cash ( $c_2$ ). Of course,  $H_{int}$  also contains the adjoint contribution, which describes the opposite situation:  $\tau_2$  sells a share to  $\tau_1$ . Therefore, for obvious reasons,  $s_j c_j^\dagger$  and  $s_j^\dagger c_j$  can be collectively called *the selling and buying operators*, respectively.  $\lambda$  is an interaction parameter: if  $\lambda = 0$ ,  $\tau_1$  and  $\tau_2$  do not interact, and we go back to our analysis in [3].

It is worth stressing that our choice of Hamiltonian is not compatible with the fact will decay and measure the Fisher information. If the domain of prices is sufficiently large, once can show indeed that the LoI increases when the value of the reservoir decreases.

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<sup>4</sup>In some older models, [5],  $c_1^\dagger$  was replaced by  $c_1^{\dagger \hat{P}}$ , where  $\hat{P}$  is the price operator.

that the amount of cash and the number of shares are preserved during the time evolution. This is a simple consequence of the fact that, calling  $\hat{K} = \sum_{j=1}^2 \hat{K}_j$  and  $\hat{S} = \sum_{j=1}^2 \hat{S}_j$  the total cash and number of shares operators of the market, they do not commute with  $H$ . In fact, in particular, they do not commute with  $H_{inf}$ :  $[H, \hat{K}] \neq 0$ ,  $[H, \hat{S}] \neq 0$ . Hence, we are allowing here for bankruptcy. Moreover, we are not assuming that the cash is only used to buy shares, so that it needs not to be preserved in time. However, some other self-adjoint operators are preserved during the time evolution. These operators are  $\hat{M}_j = \hat{S}_j + \hat{K}_j + \hat{I}_j + \hat{R}_j = \hat{\Pi}_j + \hat{I}_j + \hat{R}_j$ ,  $j = 1, 2$ , where  $\hat{R}_j = \int_{\mathbb{R}} r_j^\dagger(k) r_j(k) dk$  and  $\hat{\Pi}_j = \hat{S}_j + \hat{K}_j$  is what we call the *portfolio operator* of  $\tau_j$ , which is simply the sum of the trader  $j$ 's amount of cash and number of shares (and, being the price of each share equal to one, also their value). Then we can check that  $[H, \hat{M}_j] = 0$ ,  $j = 1, 2$ . This implies that what is constant in time is the sum of the portfolio, the LoI and the *reservoir input* of each single trader<sup>5</sup>.

What we are willing to deduce is the time evolution of the portfolio operators:  $\hat{\Pi}_j(t) = \hat{K}_j(t) + \hat{S}_j(t)$ ,  $j = 1, 2$ . As we have already noticed, because of our working assumption about the price of the shares, the mean value of  $\hat{\Pi}_j(t)$  represents for us the *richness* of  $\tau_j$ . The mean value, as widely discussed in [5], has to be taken with respect to vectors which are eigenstates of (all) the number operators of the system, with eigenvalues corresponding to the initial conditions of the system. We will see explicitly how this computation works later on.

Once we have the Hamiltonian, we can deduce the differential equations we are interested in by adopting the standard quantum mechanical Heisenberg approach:  $\dot{X} = i[H, X]$ . In this way, we get the following set of equations:

$$\begin{cases} \frac{d}{dt} s_k(t) = -i\omega_k^s s_k(t) - i\lambda_{inf} i_k(t) - i\lambda c_k(t) s_{\bar{k}}(t) c_{\bar{k}}^\dagger(t), \\ \frac{d}{dt} c_k(t) = -i\omega_k^c c_k(t) - i\lambda_{inf} i_k(t) - i\lambda s_k(t) c_{\bar{k}}(t) s_{\bar{k}}^\dagger(t), \\ \frac{d}{dt} i_k(t) = -i\Omega_k i_k(t) - i\lambda_{inf} (s_k(t) + c_k(t)) - i\gamma_k \int_{\mathbb{R}} r_k(q, t) dq \\ \frac{d}{dt} r_k(q, t) = -i\Omega_k^{(r)}(q) r_k(q, t) - i\gamma_k i_k(t), \end{cases} \quad (2.4)$$

where the *denial* of  $k$ ,  $\bar{k}$ , is seen as follows: if  $k = 1$ , then  $\bar{k} = 2$ . On the other hand, when  $k = 2$ , then  $\bar{k} = 1$ . With respect to the equations deduced in [3], in this paper we deduce

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<sup>5</sup>The existence of conserved quantities has proven to be useful, among other reasons, also to check that the numerical schemes adopted to solve the equations of the system work properly, [4].

two highly nonlinear contributions in the first two equations above. Not surprisingly, we are not able to solve the system exactly. Still, we will produce a perturbative solution which, we believe, is of some interest.

Our first step consists in rewriting the last equation in its integral form:

$$r_k(q, t) = r_k(q) e^{-i\Omega_k^{(r)}(q)t} - i\gamma_k \int_0^t i_k(t_1) e^{-i\Omega_k^{(r)}(q)(t-t_1)} dt_1,$$

and replacing this in the differential equation for  $i_k(t)$ . Assuming first that  $\Omega_k^{(r)}(q)$  is linear in  $q$ ,  $\Omega_k^{(r)}(q) = \Omega_k^{(r)} q$ , [5], we deduce that

$$\frac{d}{dt} i_k(t) = - \left( i\Omega_k + \frac{\pi\gamma_k^2}{\Omega_k^{(r)}} \right) i_k(t) - i\gamma_k \int_{\mathbb{R}} r_k(k) e^{-i\Omega_k^{(r)} q t} dq - i\lambda_{inf}(s_k(t) + c_k(t)). \quad (2.5)$$

So far, our computations are exact. However, to find some analytical solution, we are forced to consider some approximations and to perform some perturbative expansion. For this reason, as in [3], we will now work under the assumption that the last contribution in this equation can be neglected, when compared to the other ones. In other words, we are taking  $\lambda_{inf}$  to be very small. However, our procedure is much better than simply considering  $\lambda_{inf} = 0$  in  $H$  above, since we will keep memory of its effects in the first two equations in (2.4). Solving now (2.5) in its simplified expression, we get

$$i_k(t) = e^{-\left(i\Omega_k + \frac{\pi\gamma_k^2}{\Omega_k^{(r)}}\right)t} \left( i_k(0) - i\gamma_k \int_{\mathbb{R}} r_k(q) \rho_k(q, t) dq \right), \quad (2.6)$$

where

$$\rho_k(q, t) = \int_0^t e^{\left[i(\Omega_k - \Omega_k^{(r)} q) + \frac{\pi\gamma_k^2}{\Omega_k^{(r)}}\right]t_1} dt_1 = \frac{e^{\left[i(\Omega_k - \Omega_k^{(r)} q) + \frac{\pi\gamma_k^2}{\Omega_k^{(r)}}\right]t} - 1}{i(\Omega_k - \Omega_k^{(r)} q) + \frac{\pi\gamma_k^2}{\Omega_k^{(r)}}}.$$

The differential equations for  $s_j(t)$  and  $c_j(t)$  look now as follows:

$$\begin{cases} \dot{s}_1(t) = -i\omega_1^s s_1(t) - i\lambda c_1(t) s_2(t) c_2^\dagger(t) - i\lambda_{inf} i_1(t), \\ \dot{s}_2(t) = -i\omega_2^s s_2(t) - i\lambda c_2(t) s_1(t) c_1^\dagger(t) - i\lambda_{inf} i_2(t), \\ \dot{c}_1(t) = -i\omega_1^c c_1(t) - i\lambda s_1(t) c_2(t) s_2^\dagger(t) - i\lambda_{inf} i_1(t), \\ \dot{c}_2(t) = -i\omega_2^c c_2(t) - i\lambda s_2(t) c_1(t) s_1^\dagger(t) - i\lambda_{inf} i_2(t). \end{cases} \quad (2.7)$$

**Remark:** So far, the order of the various operators appearing in the right-hand side of these equations is not important since they all commute between them at equal time:  $[c_1(t), s_1^\dagger(t)] = 0$ , for all  $t \in \mathbb{R}$ , and so on. This is a consequence of the analogous commutation rule at  $t = 0$ , and of the fact that the time evolution is unitarily implemented by  $H$ :  $X(t) = e^{iHt}X(0)e^{-iHt}$ , for each dynamical variable  $X$ .

## II.1 What if we remove the information?

We devote this short subsection to briefly discuss how crucial the information really is in our model. First we check what happens if  $H_{inf} = 0$  in the definition of  $H$ . A simple computation shows that, in this case, the differential equations deduced by this new hamiltonian coincide exactly with those in (2.7), with  $i_1(t) = i_2(t) = 0$ . This shows that the presence of  $H_{inf}$  in  $H$  is to produce something like an external force driving the time evolution of the dynamical variables we are interested in,  $s_j(t)$  and  $c_j(t)$ , and  $\hat{\Pi}_j(t)$  as a consequence. It should be stressed that, in (2.7),  $i_1(t)$  and  $i_2(t)$  are now known operator-valued functions of time given in (2.6). In other words, removing  $H_{inf}$  is like removing these known forces.

Let us now look for the dynamical behavior of the two portfolio operators in this case: in principle, we should solve the Heisenberg differential equations in (2.7) putting  $\lambda_{inf} = 0$ . Needless to say, this system is not trivial, and a solution could be found when one considers a perturbation scheme for when  $\lambda$  is a small parameter. However, due to the canonical commutation rules we have assumed here, (2.2), it is easy to check that the dynamics of the two portfolios is trivial. In fact, since  $H = H_0 + H_{int}$ , it is a simple exercise to check that  $[H, \hat{\Pi}_j] = 0$ ,  $j = 1, 2$ . Hence,  $\hat{\Pi}_j(t) = \Pi_j(0)$  for all  $t \in \mathbb{R}$ : even if the cash and the shares of the two traders may change in time, their portfolios do not. This result seems reasonable since our traders, receiving no information from outside the market, have no real reason to change their original status, even if they could, in principle, interact. However, this conclusion is strongly related to the fact that, in our model, the price of the share stays constant in time. In fact, if this is not so, then the portfolio of, say,  $\tau_1$ , should be defined more reasonably as  $\Pi_1(t) := \hat{K}_1(t) + \hat{P}(t)\hat{S}_1(t)$ ,  $\hat{P}(t)$  being the

value of the share at time  $t$ , and this operator needs not to commute with  $H$ , even when  $H_{inf} = 0$ .

The conclusion of this simple analysis is therefore that, in order not to trivialize the model,  $H_{inf}$  cannot be taken to be zero, so that the equations to be solved are exactly those in (2.7), but with all their ingredients inside!

### III The perturbative solution of the equations

Our previous results suggest to check, first of all, that when  $H_{inf} \neq 0$ , the portfolio operators do not commute with  $H$ . In fact, as shown before, if they commute, there is no reason to try to solve the differential equations, and the model is (essentially) trivial, and surely not very interesting for us. However, luckily enough, this is not so:

$$[H, \hat{\Pi}_j] = \lambda_{inf} \left( i_j^\dagger (s_j + c_j) - i_j (s_j^\dagger + c_j^\dagger) \right),$$

$j = 1, 2$ . This, again, is a measure of the relevance of the information in our model: it is exactly the presence of  $H_{inf}$  which makes the model not trivial, not really the interaction between  $\tau_1$  and  $\tau_2$ .

We are now ready to set up our perturbation scheme. For that, it is convenient to define new variables  $\sigma_j(t) := s_j(t)e^{i\omega_j^s t}$  and  $\theta_j(t) := c_j(t)e^{i\omega_j^c t}$ ,  $j = 1, 2$ . To simplify the treatment a little bit, we also assume that  $\lambda = \lambda_{inf}$ . From an economical point of view, this simply means that we are assuming that the interaction and the information terms in  $H$  have a similar strength. Then equations (2.7) become

$$\begin{cases} \dot{\sigma}_1(t) = -i\lambda \left( \sigma_2(t)\theta_1(t)\theta_2^\dagger(t)e^{i\hat{\omega}t} + i_1(t)e^{i\omega_1^s t} \right), \\ \dot{\sigma}_2(t) = -i\lambda \left( \sigma_1(t)\theta_1^\dagger(t)\theta_2(t)e^{-i\hat{\omega}t} + i_2(t)e^{i\omega_2^s t} \right), \\ \dot{\theta}_1(t) = -i\lambda \left( \sigma_1(t)\sigma_2^\dagger(t)\theta_2(t)e^{-i\hat{\omega}t} + i_1(t)e^{i\omega_1^c t} \right), \\ \dot{\theta}_2(t) = -i\lambda \left( \sigma_1^\dagger(t)\sigma_2(t)\theta_1(t)e^{i\hat{\omega}t} + i_2(t)e^{i\omega_2^c t} \right), \end{cases} \quad (3.1)$$

where  $\hat{\omega} = \omega_1^s - \omega_2^s - \omega_1^c + \omega_2^c$ . The zero-th approximation in  $\lambda$  is quite simple:  $\dot{\sigma}_j^{(0)}(t) = \dot{\theta}_j^{(0)}(t) = 0$ , for  $j = 1, 2$ . Therefore, with obvious notation,  $\sigma_j^{(0)}(t) = \sigma_j^{(0)}(0) = s_j$  and

$\theta_j^{(0)}(t) = \theta_j^{(0)}(0) = c_j$ ,  $j = 1, 2$ , which we insert in the right-hand side of system (3.1) to deduce the first order approximation for  $\sigma_j(t)$  and  $\theta_j(t)$ . By introducing the new (known) operators

$$I_j^s(t) := \int_0^t i_j(t_1) e^{i\omega_j^s t_1} dt_1, \quad I_j^c(t) := \int_0^t i_j(t_1) e^{i\omega_j^c t_1} dt_1,$$

and by assuming that  $\sigma_j^{(1)}(0) = s_j$ ,  $\theta_j^{(1)}(0) = c_j$  and that  $\hat{\omega} \neq 0$ , we get

$$\begin{cases} \sigma_1^{(1)}(t) = s_1 - \frac{\lambda}{\hat{\omega}} (e^{i\hat{\omega}t} - 1) s_2 c_1 c_2^\dagger - i\lambda I_1^s(t), \\ \sigma_2^{(1)}(t) = s_2 + \frac{\lambda}{\hat{\omega}} (e^{-i\hat{\omega}t} - 1) s_1 c_1^\dagger c_2 - i\lambda I_2^s(t), \\ \theta_1^{(1)}(t) = c_1 + \frac{\lambda}{\hat{\omega}} (e^{-i\hat{\omega}t} - 1) s_1 s_2^\dagger c_2 - i\lambda I_1^c(t), \\ \theta_2^{(1)}(t) = c_2 - \frac{\lambda}{\hat{\omega}} (e^{i\hat{\omega}t} - 1) s_1^\dagger s_2 c_1 - i\lambda I_2^c(t). \end{cases} \quad (3.2)$$

It is not hard to check that this first order in our perturbation expansion is not enough: in fact, [5], in order to deduce, the (classical) function  $n_j(t)$ , we have to compute the following mean value:

$$\begin{aligned} n_j(t) &:= \left\langle \varphi_{\mathcal{G}_0}, s_j^\dagger(t) s_j(t) \varphi_{\mathcal{G}_0} \right\rangle = \left\langle \varphi_{\mathcal{G}_0}, \sigma_j^\dagger(t) \sigma_j(t) \varphi_{\mathcal{G}_0} \right\rangle \simeq \\ &\simeq \left\langle \varphi_{\mathcal{G}_0}, (\sigma_j^{(1)}(t))^\dagger \sigma_j^{(1)}(t) \varphi_{\mathcal{G}_0} \right\rangle. \end{aligned}$$

Analogously,

$$k_j(t) := \left\langle \varphi_{\mathcal{G}_0}, c_j^\dagger(t) c_j(t) \varphi_{\mathcal{G}_0} \right\rangle \simeq \left\langle \varphi_{\mathcal{G}_0}, (\theta_j^{(1)}(t))^\dagger \theta_j^{(1)}(t) \varphi_{\mathcal{G}_0} \right\rangle.$$

Here the vector  $\varphi_{\mathcal{G}_0}$  is

$$\varphi_{\mathcal{G}_0} = \frac{1}{\sqrt{n_1! n_2! k_1! k_2! I_1! I_2!}} (s_1^\dagger)^{n_1} (s_2^\dagger)^{n_2} (c_1^\dagger)^{k_1} (c_2^\dagger)^{k_2} (i_1^\dagger)^{I_1} (i_2^\dagger)^{I_2} \varphi_{\underline{0}},$$

and  $\varphi_{\underline{0}}$  is the vacuum of  $s_j$ ,  $c_j$  and  $i_j$ :  $s_j \varphi_{\underline{0}} = c_j \varphi_{\underline{0}} = i_j \varphi_{\underline{0}} = 0$ ,  $j = 1, 2$ , see [5]. The explicit choice of the numbers  $n_1$ ,  $n_2$ ,  $k_1$ ,  $k_2$ ,  $I_1$  and  $I_2$  depends on the original (i.e., at  $t = 0$ ) status of the two traders: for example,  $n_1$  is the number of share that  $\tau_1$  has at  $t = 0$ ,  $k_1$  are the units of cash in his portfolio, at this same time, while  $I_1$  is his LoI. Easy computations show that, at this order in  $\lambda$ ,  $n_j(t) = n_j(0) = n_j$  and  $k_j(t) = k_j(0) = k_j$ , so that each portfolio stays constant in time:  $\Pi_j(t) = \Pi_j(0)$ . The conclusion is therefore

that, if we want to get some non trivial dynamics, we need to go, at least, at the second order in the perturbation expansion.

This second order has to be deduced in the same way: we replace the first order solution in the right-hand side of system (3.1), and then we simply integrate on time, requiring that  $\sigma_j^{(2)}(0) = s_j$  and  $\theta_j^{(2)}(0) = c_j$ . Incidentally, we should observe that because of this approximation, we get problems of ordering of the operators. In fact, while as we have already discussed,  $\sigma_2(t)\theta_1(t)\theta_2^\dagger(t) = \theta_1(t)\sigma_2(t)\theta_2^\dagger(t) = \sigma_2(t)\theta_1(t)\theta_2^\dagger(t)$ , these equalities are false when we replace the operators with their first, or second, order approximations. For this reason we adopt here the following *normal ordering rule*: every time we have products of operators, we order them considering first  $s_1$  or  $s_1^\dagger$ , then  $s_2$  or  $s_2^\dagger$ ,  $c_1$  or  $c_1^\dagger$  and, finally,  $c_2$  or  $c_2^\dagger$ . In particular the equations in (3.1) are already written in this normal-ordered form. Needless to say, this is an arbitrary choice and needs not to be, in principle, the *best one*. Here we just want to remind that normal ordering procedures are rather common in quantum mechanics for systems with infinite degrees of freedom, and that they have proved to be quite often useful and reasonable, producing results which are in good agreement with experimental data.

After some lengthy but straightforward computations we get the following results:

$$\begin{cases} \sigma_1^{(2)}(t) = s_1 - i\lambda(-i\eta_1(t)X_1 + I_1^s(t)) - i\lambda^2(Q_1(t) + \overline{\eta_2(t)}Y_1) \\ \sigma_2^{(2)}(t) = s_2 - i\lambda(i\overline{\eta_1(t)}X_2 + I_2^s(t)) - i\lambda^2(Q_2(t) + \eta_2(t)Y_2) \\ \theta_1^{(2)}(t) = c_1 - i\lambda(i\overline{\eta_1(t)}X_3 + I_1^c(t)) - i\lambda^2(Q_3(t) + \eta_2(t)Y_3) \\ \theta_2^{(2)}(t) = c_2 - i\lambda(-i\eta_1(t)X_4 + I_2^c(t)) - i\lambda^2(Q_4(t) + \overline{\eta_2(t)}Y_4), \end{cases} \quad (3.3)$$

where we have proposed the following quantities:

$$\eta_1(t) := \frac{e^{i\hat{\omega}t} - 1}{\hat{\omega}}, \quad \eta_2(t) = \int_0^t \eta_1(t_1)e^{-i\hat{\omega}t_1}dt_1 = \frac{1}{\hat{\omega}}(t - i\overline{\eta_1(t)}),$$

$$\begin{aligned} X_1 &:= s_2c_1c_2^\dagger, & X_2 &:= s_1c_1^\dagger c_2, & X_3 &:= s_1s_2^\dagger c_2, & X_4 &:= s_1^\dagger s_2c_1, \\ Y_1 &:= s_1(c_1^\dagger c_1c_2c_2^\dagger + s_2s_2^\dagger c_2c_2^\dagger - c_1c_1^\dagger s_2s_2^\dagger), & Y_2 &:= s_2(-s_1s_1^\dagger c_1^\dagger c_1 + s_1s_1^\dagger c_2^\dagger c_2 - c_1c_1^\dagger c_2^\dagger c_2), \\ Y_3 &:= c_1(s_1s_1^\dagger c_2^\dagger c_2 - s_2s_2^\dagger c_2^\dagger c_2 - s_1s_1^\dagger s_2^\dagger s_2), & Y_4 &:= c_2(s_1^\dagger s_1s_2s_2^\dagger + s_1^\dagger s_1c_1^\dagger c_1 - s_2^\dagger s_2c_1^\dagger c_1), \end{aligned}$$



as well as the following time-dependent operators:

$$\begin{aligned}
G_1(t) &:= -i \left( -s_2 c_1 I_2^c(t)^\dagger + s_2 c_2^\dagger I_1^c(t) + c_1 c_2^\dagger I_2^s(t) \right), \\
G_2(t) &:= -i \left( s_1 c_1^\dagger I_2^c(t) - s_1 c_2 I_1^c(t)^\dagger + c_1^\dagger c_2 I_1^s(t) \right), \\
G_3(t) &:= -i \left( s_1 s_2^\dagger I_2^c(t) + s_2^\dagger c_2 I_1^s(t) - s_1 c_2 I_2^s(t)^\dagger \right), \\
G_4(t) &:= -i \left( s_1^\dagger s_2 I_1^c(t) + s_1^\dagger c_1 I_2^s(t) - s_2 c_1 I_1^s(t)^\dagger \right),
\end{aligned}$$

and

$$Q_j(t) := \begin{cases} \int_0^t G_j(t_1) e^{i\hat{\omega}t_1} dt_1, & j = 1, 4, \\ \int_0^t G_j(t_1) e^{-i\hat{\omega}t_1} dt_1, & j = 2, 3. \end{cases}$$

We can now compute the mean values of  $\sigma_j^{(2)\dagger}(t)\sigma_j^{(2)}(t)$  and  $\theta_j^{(2)\dagger}(t)\theta_j^{(2)}(t)$  on the state  $\langle \varphi_{\mathcal{G}_0}, \cdot \varphi_{\mathcal{G}_0} \rangle$  as seen before. Another approximation is adopted at this stage: formula (2.6) shows that the contribution of the reservoir,  $i\gamma_k \int_{\mathbb{R}} r_k(q) \rho_k(q, t) dq$ , is  $O(\gamma_k)$  with respect to the other contribution,  $i_k(0)$ . For this reason, assuming  $\gamma_k$  to be small enough, we approximate  $i_k(t)$  with  $e^{-\left(i\Omega_k + \frac{\pi\gamma_k^2}{\Omega_k^{(r)}}\right)t} i_k(0)$ .

Up to the second order in  $\lambda$ , we get

$$\begin{aligned}
n_1(t) &\simeq n_1 + \frac{2\lambda^2}{\hat{\omega}^2} (1 - \cos(\hat{\omega}t)) [n_1(k_1 n_2 - k_1 k_2 - n_2 k_2 - k_2) + n_2 k_1 (1 + k_2)] + \lambda^2 I_1 |\eta_3^s(t)|^2, \\
n_2(t) &\simeq n_2 + \frac{2\lambda^2}{\hat{\omega}^2} (1 - \cos(\hat{\omega}t)) [n_2(n_1 k_2 - k_1 k_2 - k_1 n_1 - k_1) + n_1 k_2 (1 + k_1)] + \lambda^2 I_2 |\eta_4^s(t)|^2, \\
k_1(t) &\simeq k_1 + \frac{2\lambda^2}{\hat{\omega}^2} (1 - \cos(\hat{\omega}t)) [k_1(n_1 k_2 - n_1 n_2 - n_2 k_2 - n_2) + n_1 k_2 (1 + n_2)] + \lambda^2 I_1 |\eta_3^c(t)|^2, \\
k_2(t) &\simeq k_2 + \frac{2\lambda^2}{\hat{\omega}^2} (1 - \cos(\hat{\omega}t)) [k_2(k_1 n_2 - n_1 n_2 - n_1 k_1 - n_1) + k_1 n_2 (1 + n_1)] + \lambda^2 I_2 |\eta_4^c(t)|^2.
\end{aligned}$$

Incidentally, these results confirm that the first non trivial contribution in our perturbation scheme is quadratic in  $\lambda$ . The following quantity has been proposed:

$$\eta_k^s(t) = \int_0^t e^{\left[i(\omega_{k-2}^s - \Omega_{k-2}) - \frac{\pi\gamma_{k-2}^2}{\Omega_{k-2}^{(r)}}\right]t_1} dt_1 = \frac{e^{\left[i(\omega_{k-2}^s - \Omega_{k-2}) - \frac{\pi\gamma_{k-2}^2}{\Omega_{k-2}^{(r)}}\right]t} - 1}{i(\omega_{k-2}^s - \Omega_{k-2}) - \frac{\pi\gamma_{k-2}^2}{\Omega_{k-2}^{(r)}}},$$

for  $k = 3, 4$ . The other function  $\eta_k^c(t)$ , is defined like  $\eta_k^s(t)$  with the only difference that  $\omega_{k-2}^s$  is replaced by  $\omega_{k-2}^c$ . If we now compute the variation of the portfolios,  $\delta\Pi_j(t) := \Pi_j(t) - \Pi_j(0)$ , we find that

$$\delta\Pi_1(t) = \lambda^2 I_1 (|\eta_3^s(t)|^2 + |\eta_3^c(t)|^2), \quad \delta\Pi_2(t) = \lambda^2 I_2 (|\eta_4^s(t)|^2 + |\eta_4^c(t)|^2). \quad (3.4)$$

These formulas show, first of all, that up to the order  $\lambda^2$ , what is really important in the computation of the portfolios of the traders, is not the initial conditions on the cash and shares but, much more than this, the initial values of the LoI for each trader<sup>6</sup>. This is the only *quantum number* which appears in (3.4), while all the other numbers,  $n_1, n_2, k_1$  and  $k_2$ , produce contributions which sum up to zero, at least at this order in  $\lambda$ .

Another interesting feature of the analytical expressions for  $\delta\Pi_j(t)$  can be deduced observing that,

$$|\eta_3^s(t)|^2 = \frac{e^{-\frac{2\pi\gamma_1^2}{\Omega_1^{(r)}}t} - 2e^{-\frac{\pi\gamma_1^2}{\Omega_1^{(r)}}t} \cos(\omega_1^s - \Omega_1)t + 1}{(\omega_1^s - \Omega_1)^2 + \frac{\pi^2\gamma_1^4}{\Omega_1^{(r)^2}}}.$$

This implies that  $\delta\Pi_1(t)$  and  $\delta\Pi_2(t)$  both admit a non trivial asymptotic value: calling  $\delta\Pi_j(\infty) = \lim_{t \rightarrow \infty} \delta\Pi_j(t)$ , and using the above formula for  $|\eta_3^s(t)|^2$  and the analogous formulas for  $|\eta_3^c(t)|^2$ ,  $|\eta_4^s(t)|^2$  and  $|\eta_4^c(t)|^2$ , we get

$$\delta\Pi_1(\infty) = \lambda^2 I_1 \left( \frac{1}{(\omega_1^s - \Omega_1)^2 + \frac{\pi^2\gamma_1^4}{\Omega_1^{(r)^2}}} + \frac{1}{(\omega_1^c - \Omega_1)^2 + \frac{\pi^2\gamma_1^4}{\Omega_1^{(r)^2}}} \right), \quad (3.5)$$

and

$$\delta\Pi_2(\infty) = \lambda^2 I_2 \left( \frac{1}{(\omega_2^s - \Omega_2)^2 + \frac{\pi^2\gamma_2^4}{\Omega_2^{(r)^2}}} + \frac{1}{(\omega_2^c - \Omega_2)^2 + \frac{\pi^2\gamma_2^4}{\Omega_2^{(r)^2}}} \right), \quad (3.6)$$

The first evident conclusion is that  $\delta\Pi_1(\infty) + \delta\Pi_2(\infty) \neq 0$ . This is possible, since the total amount of cash and the total number of shares are not required to be constant in time, in our model. Therefore, there is no reason to expect that the gain for  $\tau_1$  become the loss for  $\tau_2$ , or viceversa.

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<sup>6</sup>If we consider [16], the level of LoI can depend i) on how large the domain of prices of the payoff function is; ii) the type of payoff function and iii) the level of public information.

As we can see, in agreement with our general analysis in [5], the parameters of the free Hamiltonian behave as a sort of inertia for the system. More in details, if  $\omega_1^s$  and  $\omega_1^c$  are very large, compared with  $\Omega_1$  and  $\Omega_2$ , we see that  $\delta\Pi_1(\infty)$  is very small:  $\tau_1$  experiences a large inertia, so that the value of his portfolio stays almost constant. A similar conclusion is deduced if  $\frac{\gamma_1^2}{\Omega_1^{(r)}}$  is large enough. Let us now suppose that  $\omega_1^s = \omega_1^c = \Omega_1$ . Then  $\delta\Pi_1(\infty) = 2\lambda^2 I_1 \frac{\Omega_1^{(r)2}}{\pi^2 \gamma_1^4}$ . We see from this formula that the reservoir of the information plays also a role in the evolution of the portfolios, and we see that, what is relevant for us, is not really the contribution of the free Hamiltonian,  $\Omega_1^{(r)}$ , or the contribution of the interaction between the reservoir and the dynamical variables of the LoI,  $\gamma_1$ , but the ratio above between the two. This is interesting because it shows that we do have a contribution to  $\delta\Pi_j(\infty)$  coming from these parts of the full Hamiltonian, even under all the approximations we have considered along the way.

On the other hand, for  $\delta\Pi_1(\infty)$  to be large, it is convenient to have large  $I_1$  and/or small values of  $\omega_1^s - \Omega_1$ ,  $\omega_1^c - \Omega_1$  and of  $\frac{\gamma_1^2}{\Omega_1^{(r)}}$ . Similar conclusions can be deduced for  $\delta\Pi_2(\infty)$ .

A natural question is the following: when does it happen that  $\delta\Pi_1(\infty) > \delta\Pi_2(\infty)$ ? This is ensured, for sure, if all the following inequalities are satisfied:

$$I_1 > I_2, \quad \omega_1^s - \Omega_1 < \omega_2^s - \Omega_2, \quad \omega_1^c - \Omega_1 < \omega_2^c - \Omega_2, \quad \frac{\gamma_1^2}{\Omega_1^{(r)}} < \frac{\gamma_2^2}{\Omega_2^{(r)}}.$$

Particularly interesting is what happens if  $\omega_1^s - \Omega_1 = \omega_2^s - \Omega_2$  and  $\omega_1^c - \Omega_1 = \omega_2^c - \Omega_2$ . In this case, in order to have  $\delta\Pi_1(\infty) > \delta\Pi_2(\infty)$ , we need to compare two ingredients of the formulas, i.e.  $I_j$  and the ratio  $\frac{\gamma_j^2}{\Omega_j^{(r)}}$ . As we have seen before, in these conditions

$\delta\Pi_1(\infty) > \delta\Pi_2(\infty)$  surely if  $I_1 > I_2$  and if  $\frac{\gamma_1^2}{\Omega_1^{(r)}} < \frac{\gamma_2^2}{\Omega_2^{(r)}}$ . But the first inequality implies that the LoI of  $\tau_1$  should be larger than that of  $\tau_2$ , while the second inequality can be rewritten as  $\frac{\Omega_2^{(r)}}{\gamma_2^2} < \frac{\Omega_1^{(r)}}{\gamma_1^2}$ . This suggests to divide the information reaching the traders in two different kinds: a *bad information*, which is directly related to the variables  $i_j$ ,  $i_j^\dagger$  and  $\hat{I}_j$ , and a *good one*<sup>7</sup>, which is related to the reservoir and, therefore, to the variables  $r_j(q)$ ,  $r_j^\dagger(q)$  and  $\hat{R}_j(q)$ . This is an interesting result, since it helps to clarify the roles of

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<sup>7</sup>If we consider [16], the total energy, if it is the harbinger of public information (relative thus to the payoff function), it will not necessarily be classified as bad or good information for the portfolio holder.

the different ingredients of the Hamiltonian (2.1). The differentiation of information into ‘good’ and ‘bad’ information can also be found back in early work in finance. The so called ‘Kyle measure’ [25] was proposed to give an indication of how the level of private information compares to the level of so called noise trading (which itself is based on a type of information which is different from private information).

## IV Conclusions

In this paper we have discussed, within an operatorial setting, a simple stock market formed by just two traders who, whilst they are interacting between them, are subjected to a flux of information which aids them to decide how to behave during the trade operations. A non perturbative result shows that, in order to not get trivial dynamics, we need to put information in the model. Otherwise the portfolios of the traders do not change in time. Using a perturbation expansion we have also deduced the time evolution of the portfolios of the two traders and we have analyzed their asymptotic limits at a second order in perturbation theory. This analysis suggests to contemplate a difference between a bad and a good information. We believe that this is quite a natural distinction, and it clarifies the meaning of the various terms in  $H$ . Interestingly enough, the bad information is related to a set of two-modes bosonic operators, while the good information arises from two reservoirs, each having an infinite number of modes.

Needless to say, a step toward real models would imply the following improvements: more traders, different kind of shares and non constant prices of the shares. Although the first two extensions do not look particularly difficult, the last one is very complicated. We hope to be able to produce such a model in the near future.

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The diminishing of public information may affect the level of private information in a different way, if the domain of the payoff function is small, as opposed to the case when the domain of the payoff function is large.

## Acknowledgements

F.B. acknowledges partial financial support from Università di Palermo. F.B. also wishes to thank the School of Management and Institute of Finance of the University of Leicester for its warm hospitality.

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